⑤

Since we use ordered pairs, in general $(a,b) \neq (b,a)$ so $\boxed{A\times B \neq B\times A}$

(though there is a sense of sameness)

Ex: Write out the set $A\times B$ where $A=\{1,2,3,4\}$ and $B=\{a,c\}$.

Sol:

$$A\times B=\{(1,a),(1,c),(2,a),(2,c),(3,a),(3,c),(4,a),(4,c)\}$$

Fact: For finite sets A and B

$$|A\times B| = |A|\cdot|B|$$

(this is also true for infinite sets, but one needs to understand what $|A|$ means there and how to multiply infinite cardinals)

We don't have to stop at taking the product of two sets. For example:

$$A\times(B\times C)=\{(a,(b,c)) \mid a\in A, b\in B, c\in C\}$$

Similar to the reasons above

$$\boxed{A\times(B\times C) \neq (A\times B)\times C}$$

⑥

Though, again, there is a sense of sameness.

We can also form the product of 3 sets by using ordered triples instead of nested ordered pairs:

$$A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$$

Once again,

$$A \times B \times C \neq A \times (B \times C) \neq (A \times B) \times C$$

but there is a sense of sameness (they're in one-to-one correspondence).

This lead us to the notion of Cartesian powers of sets: for a set A and $k \in \mathbb{N}$

$$A^k := \{(a_1, a_2, \cdots, a_k) \mid a_i \in A, 1 \leq i \leq k\}$$

Ex: If we write the "package" of numbers in a different way we can see that 3×3 matrices are very similar to 9-tuples

$$M_{3\times3}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \middle| a,b,c,d,e,f,g,h,j \in \mathbb{R} \right\}$$

$$\mathbb{R}^9 = \{(a,b,c,d,e,f,g,h,j) \mid a,b,c,d,e,f,g,h,j \in \mathbb{R}\}$$

I still insist $M_{3\times3}(\mathbb{R}) \neq \mathbb{R}^9$, but their likeness is unmistakable. This relationship is useful for proving facts about 3x3 matrices, though we will not see them here.

1.3 - Subsets

Suppose that A and B are sets, and that every element in A is also in B. Then we say that A is a subset of B. We denote this by

$$\boxed{A \subset B}.$$

If A is not a subset of B, we write

$$\boxed{A \not\subset B}.$$

This would be the case if A has even one element which is not in B.

Ex
(a) $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
(b) $\{2,3,5,7,11,\cdots\} \subset \mathbb{Z}$
(c) $\{1,2,3,5,7,11,\cdots\} \not\subset \{p \mid p \text{ is prime}\}$
(d) $\{(x,y) \mid x^2+y^2 > 3\} \subset \mathbb{R}^2$
(e) $\{2^n \mid n \in \mathbb{Z}\} \subset \mathbb{Z}$
(f) $\{2^n \mid n \in \mathbb{Z}\} \not\subset \mathbb{N}$

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<u>Fact</u>: The empty set is a subset of <u>every</u> set, i.e., for any set S

$$\emptyset \subset S$$

An interesting and useful idea is to count the number of subsets a set has.

<u>Ex</u>: List the subsets of $A=\{1,2,3\}$. How many subsets does A have?

<u>Subsets</u>: $\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}=A$

We have, then, 8 subsets. What's the relation between 3 & 8? $8=2^3$

For a set with n elements, one might then guess that it has 2^n elements. This is a true statement, but why? The reasoning is as follows:

Let $A=\{a_1, \ldots, a_n\}$ be a set with n elements. Think about constructing a subset $S\subset A$... for every element a_i we have a choice of whether to include it or not; 2 choices for each element.

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So, if we start with a_1 and go through the elements one-by-one we have:

1) Include a_1? 2 sets

2) Include a_2? We have this choice for each of the 2 sets from step 1, so we have now: $2 \times 2 = 2^2$ sets

3) For each set in step 2, decide whether to include a_3. This gives $2^2 \times 2 = 2^3$ sets

$\vdots$

n) Include a_n? 2^n sets

1.4 - Power Sets

Def: Given a set A, the set of all subsets of A, denoted $\mathcal{P}(A)$, is called the power set of A. I.e., $\boxed{\mathcal{P}(A) = \{X \mid X \subseteq A\}}$

Ex: As we saw earlier,

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Fact: Using what we saw before, if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.

Ex: Let $A = \{1, \{1,2\}\}$.

(a) What is $|\mathcal{P}(A)|$?

(b) What is $|\mathcal{P}(\mathcal{P}(A))|$?

(c) Write out $\mathcal{P}(A)$.

Sol: (a) $|A| = 2$ b/c the elements of A are 1 and $\{1,2\}$.

So $|\mathcal{P}(A)| = 2^2 = 4$

(b) $|\mathcal{P}(\mathcal{P}(A))| = 2^4 = 8$

(c) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{\{1,2\}\}, \{1, \{1,2\}\}\}$

As you can see, power sets get large exponentially quick.

What about $\mathcal{P}(\mathbb{N})$? How large is this? It actually turns out that

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$$

which is incredibly large!

What kinds of things are in $\mathcal{P}(\mathbb{R}^2)$?